Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

International Journal of Solids and Structures 42 (2005) 3321–3337

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Mode III crack problems for two bonded functionally graded piezoelectric materials

Ching-Hwei Chue *, Yi-Liang Ou

Department of Mechanical Engineering, National Cheng Kung University, No. 1, Da-Syue Road, 70101 Tainan, Taiwan, Republic of China

Received 26 June 2004; received in revised form 8 October 2004
Available online 18 November 2004

Abstract

This paper investigates the singular electromechanical field near the crack tips of an internal crack. The crack is perpendicular to the interface formed by bonding two half planes of different functionally graded piezoelectric material. The properties of two materials, such as elastic modulus, piezoelectric constant and dielectric constant, are assumed in exponential forms and vary along the crack direction. The singular integral equations for impermeable and permeable cracks are derived and solved by using the Gauss–Chebyshev integration technique. It shows that the stresses and electrical displacements around the crack tips have the conventional square root singularity. The stress intensity and electric displacement intensity factors are highly affected by the material nonhomogeneity parameters β and γ. The solutions for some degenerated problems can also be obtained.
© 2004 Elsevier Ltd. All rights reserved.

Keywords: Functionally graded piezoelectric material; Impermeable and permeable crack problems; Singular integral equation; Gauss–Chebyshev integration technique

1. Introduction

In designing a smart structure, the piezoelectric elements are often bonded to metallic or composite materials. Bimorph or multilayer piezoelectric actuators are these kinds of structures (Tomio, 1990; He and Ye, 2000). The stress peaks will be induced at the interfaces to cause failure such as cracking or debonding.

* Corresponding author. Tel.: +886 6 2757575x62165; fax: +886 6 2363950.
E-mail address: chchue@mail.ncku.edu.tw (C.-H. Chue).

0020-7683/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2004.10.013

The studies on the development of functionally graded elastic material become gradually mature. Typical papers include Delale and Erdogan (1983), Erdogan (1985), Erdogan et al. (1991a,b), Choi (1996), and Asfar and Sekine (2000).

Recently, some researchers start to investigate the application of piezoelectric material with continuously varying material properties (Zhu et al., 1995; Wu et al., 1996). The new materials are called functionally graded piezoelectric material (FGPM). To our knowledge, Li and Weng (2002) first applied the concept of fracture mechanics on a finite crack in a strip of functionally graded piezoelectric material. They found that the singular stresses and electrical displacements at the tip of the crack in the FGPM carry the same forms as those in a homogeneous piezoelectric material but the magnitudes of the intensity factors are dependent on the gradient of the FGPM properties. Wang (2003) solved the antiplane crack and collinear crack problems in FGPM. A class of functional forms has been assumed to describe the mechanical and the electrical properties of the medium. For the permeable crack, the stress and the electrical displacement intensity factors depend only on the applied mechanical loads. The piezoelectric effect has no effect on the stress intensity factors. Ueda (2003) obtained the solutions for a crack in FGPM strip bonded to two elastic surface layers. He used the energy density factors to predict the fracture behavior of the structure.

In this paper, we use the model of Erdogan (1985) and change the material to FGPM. The crack problem can be reduced into a system of singular integral equations after applying the Fourier Transform and solved numerically by using Gauss–Chebyshev integration technique. The stress and electrical displacement intensity factors are then obtained from the near crack tip field solution of electro-mechanical results.

2. Formulation of the problem

Fig. 1 shows two functionally graded piezoelectric materials perfectly bonded together along y-axis. A crack of length $2a_0$ lies in $a \leqq x \leqq b$ and perpendicular to the interface. Since the poling directions of piezoelectric materials are orientated along z-axis, the antiplane mechanical field and inplane electrical field are coupled. The constitutive equations can be written as

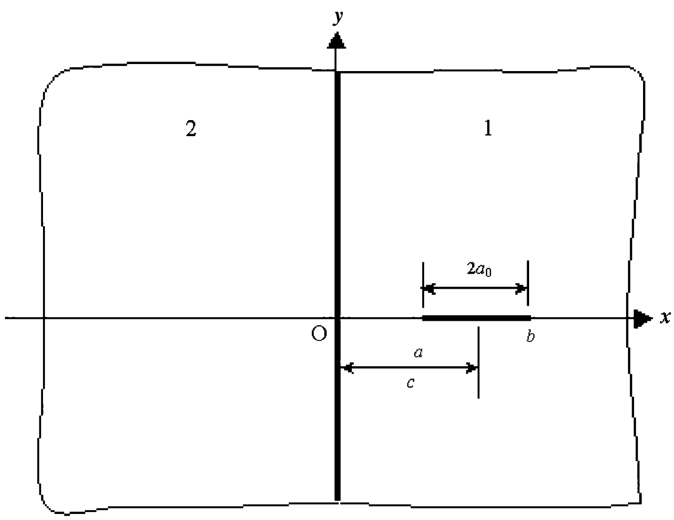

Fig. 1. Geometry of two bonded functionally graded piezoelectric materials containing a crack perpendicular to the interface.

$$\tau_{xz(i)} = c_{44(i)}(x)\frac{\partial w_i}{\partial x} + e_{15(i)}(x)\frac{\partial \phi_i}{\partial x}, \quad \tau_{yz(i)} = c_{44(i)}(x)\frac{\partial w_i}{\partial y} + e_{15(i)}(x)\frac{\partial \phi_i}{\partial y}$$
$$D_{x(i)} = e_{15(i)}(x)\frac{\partial w_i}{\partial x} - \varepsilon_{11(i)}(x)\frac{\partial \phi_i}{\partial x}, \quad D_{y(i)} = e_{15(i)}(x)\frac{\partial w_i}{\partial y} - \varepsilon_{11(i)}(x)\frac{\partial \phi_i}{\partial y} \tag{1}$$

where τ_{ij}, w_i, D_i, and ϕ_i are the shear stresses, antiplane displacements, inplane electrical displacements and electric potentials, respectively. The variations of material constants $c_{44(i)}(x)$, $e_{15(i)}(x)$, $\varepsilon_{11(i)}(x)$ called the shear moduli, piezoelectric constants, and dielectric constants, respectively, are assumed in the following exponential forms:

$$c_{44(1)}(x) = c_0 \exp(\beta x), \quad e_{15(1)}(x) = e_0 \exp(\beta x), \quad \varepsilon_{11(1)}(x) = \varepsilon_0 \exp(\beta x) \quad \text{for } x > 0$$
$$c_{44(2)}(x) = c_0 \exp(\gamma x), \quad e_{15(2)}(x) = e_0 \exp(\gamma x), \quad \varepsilon_{11(2)}(x) = \varepsilon_0 \exp(\gamma x) \quad \text{for } x < 0 \tag{2}$$

where β and γ are called nonhomogeneous parameters. The structure with negative β and positive γ indicates that two FGPMs are bonded on the stiffer sides. The constants c_0, e_0, and ε_0 are the material properties at interface.

The static equilibrium equation and Maxwell's equation under electro-static condition are given as

$$\frac{\partial \tau_{xz(i)}}{\partial x} + \frac{\partial \tau_{yz(i)}}{\partial y} = 0, \quad \frac{\partial D_{x(i)}}{\partial x} + \frac{\partial D_{y(i)}}{\partial y} = 0 \tag{3}$$

where the body forces and free charges have been neglected.

Substituting Eq. (1) into Eq. (3) and using the relations (2), we obtain the following equations for materials 1 and 2, respectively:

$$\begin{cases} c_0\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right) + e_0\left(\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}\right) + \beta\left(c_0\frac{\partial w_1}{\partial x} + e_0\frac{\partial \phi_1}{\partial x}\right) = 0 \\ e_0\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right) - \varepsilon_0\left(\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}\right) + \beta\left(e_0\frac{\partial w_1}{\partial x} - \varepsilon_0\frac{\partial \phi_1}{\partial x}\right) = 0 \end{cases} \tag{4a}$$

$$\begin{cases} c_0\left(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\right) + e_0\left(\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2}\right) + \gamma\left(c_0\frac{\partial w_2}{\partial x} + e_0\frac{\partial \phi_2}{\partial x}\right) = 0 \\ e_0\left(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\right) - \varepsilon_0\left(\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2}\right) + \gamma\left(e_0\frac{\partial w_2}{\partial x} - \varepsilon_0\frac{\partial \phi_2}{\partial x}\right) = 0 \end{cases} \tag{4b}$$

Expressing the solutions of Eqs. (4) in the following forms (Erdogan, 1985):

$$\begin{cases} w_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{11}(\alpha,y)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha + \frac{2}{\pi}\int_0^{\infty} g_{11}(x,\alpha)\sin(\alpha y)\,\mathrm{d}\alpha \\ \phi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{21}(\alpha,y)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha + \frac{2}{\pi}\int_0^{\infty} g_{21}(x,\alpha)\sin(\alpha y)\,\mathrm{d}\alpha \end{cases} \tag{5a}$$

$$\begin{cases} w_2(x,y) = \frac{2}{\pi}\int_0^{\infty} g_{12}(x,\alpha)\sin(\alpha y)\,\mathrm{d}\alpha \\ \phi_2(x,y) = \frac{2}{\pi}\int_0^{\infty} g_{22}(x,\alpha)\sin(\alpha y)\,\mathrm{d}\alpha \end{cases} \tag{5b}$$

we obtain:

$$\begin{cases} f_{11}(\alpha, y) = A_1(\alpha)\exp\left(-\sqrt{\alpha^2 + \mathrm{i}\alpha\beta}y\right) \\ f_{21}(\alpha, y) = B_1(\alpha)\exp\left(-\sqrt{\alpha^2 + \mathrm{i}\alpha\beta}y\right) \\ g_{11}(x, \alpha) = C_1(\alpha)\exp\left(\dfrac{-\beta - \sqrt{\beta^2 + 4\alpha^2}}{2}x\right) \\ g_{21}(x, \alpha) = D_1(\alpha)\exp\left(\dfrac{-\beta - \sqrt{\beta^2 + 4\alpha^2}}{2}x\right) \end{cases} \tag{6}$$

$$\begin{cases} g_{12}(x, \alpha) = E_2(\alpha)\exp\left(\dfrac{-\gamma + \sqrt{\gamma^2 + 4\alpha^2}}{2}x\right) \\ g_{22}(x, \alpha) = F_2(\alpha)\exp\left(\dfrac{-\gamma + \sqrt{\gamma^2 + 4\alpha^2}}{2}x\right) \end{cases} \tag{7}$$

The solutions of undetermined functions $A_1(\alpha)$, $B_1(\alpha)$, $C_1(\alpha)$, $D_1(\alpha)$, $E_2(\alpha)$, and $F_2(\alpha)$ depends on the mechanical and electrical conditions of crack surfaces. The electrical condition on the crack surfaces may be permeable or impermeable.

2.1. Impermeable crack problem

(i) *Continuity conditions along the interface:*

$$w_1(0, y) = w_2(0, y) \tag{8a}$$

$$\phi_1(0, y) = \phi_2(0, y) \tag{8b}$$

$$\tau_{xz(1)}(0, y) = \tau_{xz(2)}(0, y) \tag{8c}$$

$$D_{x(1)}(0, y) = D_{x(2)}(0, y) \tag{8d}$$

(ii) *Symmetric conditions:*

Since the electro-mechanical field is symmetric with respect to the x-axis, it is sufficient to consider the upper surface for $y \geqq 0$. Then, we have

$$w_1(x, 0) = 0 \quad \text{for } 0 \leqq x < a \text{ and } b < x < \infty \tag{8e}$$

$$\phi_1(x, 0) = 0 \quad \text{for } 0 \leqq x < a \text{ and } b < x < \infty \tag{8f}$$

$$w_2(x, 0) = 0 \quad \text{for } -\infty \leqq x < 0 \tag{8g}$$

$$\phi_2(x, 0) = 0 \quad \text{for } -\infty \leqq x < 0 \tag{8h}$$

Note that the assumed forms of w_2 and ϕ_2 in Eq. (5b) automatically satisfy the symmetric conditions (8g) and (8h).

(iii) *Conditions on the crack surfaces:*

The crack surface is impermeable and is simultaneously subjected to electrical displacement $D(x)$ and shear traction $\tau(x)$:

$$\tau_{yz(1)}(x,0) = \tau(x) \quad \text{for } a < x < b \tag{8i}$$

$$D_{y(1)}(x,0) = D(x) \quad \text{for } a < x < b \tag{8j}$$

where $D(x)$ and $\tau(x)$ can be obtained by using superposition method from the remote electrical and mechanical loads.

After applying the continuity conditions Eqs. (8a)–(8d) and taking Fourier inverse transform, four unknown functions can be expressed by the rest of two functions $A_1(\alpha)$ and $B_1(\alpha)$:

$$\begin{cases} E_2(\alpha) - C_1(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\alpha}{\alpha^2+\rho^2+\mathrm{i}\beta\rho}\right)A_1(\rho)\,\mathrm{d}\rho \\ F_2(\alpha) - D_1(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\alpha}{\alpha^2+\rho^2+\mathrm{i}\beta\rho}\right)B_1(\rho)\,\mathrm{d}\rho \\ c_0sE_2(\alpha) + e_0sF_2(\alpha) - c_0pC_1(\alpha) - e_0pD_1(\alpha) = \dfrac{-1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left(\frac{\alpha}{\alpha^2+\rho^2+\mathrm{i}\beta\rho}\right)\mathrm{i}\rho(c_0A_1(\rho)+e_0B_1(\rho))\,\mathrm{d}\rho \\ e_0sE_2(\alpha) - \varepsilon_0sF_2(\alpha) - e_0pC_1(\alpha) + \varepsilon_0pD_1(\alpha) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left(\frac{\alpha}{\alpha^2+\rho^2+\mathrm{i}\beta\rho}\right)\mathrm{i}\rho(-e_0A_1(\rho)+\varepsilon_0B_1(\rho))\,\mathrm{d}\rho \end{cases} \tag{9}$$

where $p = \frac{-\beta-\sqrt{\beta^2+4\alpha^2}}{2}$, $s = \frac{-\gamma+\sqrt{\gamma^2+4\alpha^2}}{2}$.

Define two dislocation functions (Erdogan, 1985) $g_1(x)$ and $g_2(x)$ as

$$g_1(x) = \frac{\partial}{\partial x}w_1(x,0) \tag{10a}$$

$$g_2(x) = \frac{\partial}{\partial x}\phi_1(x,0) \tag{10b}$$

Substituting Eqs. (5a) and (6) into Eqs. (10a) and (10b), and applying the conditions (8e) and (8f), we find that $g_1(x)$ and $g_2(x)$ must satisfy the following equations:

$$\int_a^b g_1(t)\,\mathrm{d}t = \int_a^b g_2(t)\,\mathrm{d}t = 0 \tag{11}$$

The two remaining unknown functions can be solved as

$$A_1(\alpha) = \frac{\mathrm{i}}{\alpha}\int_a^b g_1(t)\mathrm{e}^{\mathrm{i}\alpha t}\,\mathrm{d}t \tag{12a}$$

$$B_1(\alpha) = \frac{\mathrm{i}}{\alpha}\int_a^b g_2(t)\mathrm{e}^{\mathrm{i}\alpha t}\,\mathrm{d}t \tag{12b}$$

By using the residue theorem, $C_1(\alpha)$, $D_1(\alpha)$, $E_2(\alpha)$, and $F_2(\alpha)$ can be obtained:

$$C_1(\alpha) = \frac{(s-n_1)\alpha}{2n_1(p-s)\alpha_1}\int_a^b g_1(t)\mathrm{e}^{-n_1t}\,\mathrm{d}t \tag{13}$$

$$D_1(\alpha) = \frac{(s-n_1)\alpha}{2n_1(p-s)\alpha_1}\int_a^b g_2(t)\mathrm{e}^{-n_1t}\,\mathrm{d}t \tag{14}$$

$$E_2(\alpha) = \frac{(p-n_1)\alpha}{2n_1(p-s)\alpha_1}\int_a^b g_1(t)\mathrm{e}^{-n_1t}\,\mathrm{d}t \tag{15}$$

$$F_2(\alpha) = \frac{(p-n_1)\alpha}{2n_1(p-s)\alpha_1}\int_a^b g_2(t)\mathrm{e}^{-n_1 t}\,\mathrm{d}t \tag{16}$$

where $n_1 = \alpha_1 - \beta/2$, and $\alpha_1 = \sqrt{\alpha^2 + \beta^2/4}$. From Eqs. (5a), (6), (12), (13)–(16), conditions (8i) and (8j) become

$$\tau_{yz(1)}(x,0) = \tau(x) = c_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b [k_1(x,t)+k_2(x,t)]g_1(t)\,\mathrm{d}t + e_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b [k_1(x,t)+k_2(x,t)]g_2(t)\,\mathrm{d}t \tag{17}$$

$$D_{y(1)}(x,0) = D(x) = e_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b [k_1(x,t)+k_2(x,t)]g_1(t)\,\mathrm{d}t - \varepsilon_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b [k_1(x,t)+k_2(x,t)]g_2(t)\,\mathrm{d}t \tag{18}$$

where

$$k_1(x,t) = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\frac{-\sqrt{\alpha^2+\mathrm{i}\beta\alpha}}{\alpha}\mathrm{e}^{\mathrm{i}\alpha(t-x)}\,\mathrm{d}\alpha \tag{19}$$

$$k_2(x,t) = \mathrm{e}^{\frac{\beta(t-x)}{2}}\int_0^{\infty}\frac{\alpha^2(s-n_1)}{(p-s)n_1\sqrt{\alpha^2+\beta^2/4}}\mathrm{e}^{-(t+x)\sqrt{\alpha^2+\beta^2/4}}\,\mathrm{d}\alpha \tag{20}$$

The first integral of Eq. (17) is related to the FGM problem solved by Erdogan (1985). We define a function $K_2(\alpha)$ as the factor in the integrand of Eq. (20):

$$K_2(\alpha) = \frac{\alpha^2(s-n_1)}{(p-s)n_1\sqrt{\alpha^2+\beta^2/4}} = \frac{\alpha^2\left(\alpha_1-\alpha_2+\frac{\gamma-\beta}{2}\right)}{\alpha_1\left(\alpha_2+\alpha_1+\frac{\beta-\gamma}{2}\right)\left(\alpha_1-\frac{\beta}{2}\right)} \tag{21}$$

where $\alpha_2 = \sqrt{\alpha^2+\gamma^2/4}$.

Separating the singular term of the kernels $k_1(x,t)$, Eqs. (17) and (18) may be rewritten as following:

$$\begin{aligned}\tau_{yz(1)}(x,0) &= \tau(x)\\ &= c_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)\right]g_1(t)\,\mathrm{d}t\\ &\quad + e_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)\right]g_2(t)\,\mathrm{d}t\end{aligned} \tag{22}$$

$$\begin{aligned}D_{y(1)}(x,0) &= D(x)\\ &= e_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)\right]g_1(t)\,\mathrm{d}t\\ &\quad - \varepsilon_0\mathrm{e}^{\beta x}\frac{1}{\pi}\int_a^b\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)\right]g_2(t)\,\mathrm{d}t\end{aligned} \tag{23}$$

where the kernel $h_1(x,t)$ is as follows:

$$\begin{aligned}h_1(x,t) &= \int_0^{\infty}\left[\left(1+\frac{\beta^2}{\alpha^2}\right)^{0.25}\cos\left(\frac{\theta}{2}\right)-1\right]\sin\alpha(t-x)\,\mathrm{d}\alpha + \int_0^{A}\left[\left(1+\frac{\beta^2}{\alpha^2}\right)^{0.25}\sin\left(\frac{\theta}{2}\right)\right]\cos\alpha(t-x)\,\mathrm{d}\alpha\\ &\quad + \int_A^{\infty}\left[\left(1+\frac{\beta^2}{\alpha^2}\right)^{0.25}\sin\left(\frac{\theta}{2}\right)-\frac{\beta}{2\alpha}\right]\cos\alpha(t-x)\,\mathrm{d}\alpha + \frac{\beta}{2}\int_A^{\infty}\frac{\cos\alpha(t-x)}{\alpha}\,\mathrm{d}\alpha\end{aligned} \tag{24}$$

with $\tan\theta = \beta/\alpha$ and A is an arbitrary positive constant. Note that if the piezoelectric effect is ignored, the first integral of Eq. (22) is equivalent to the case of a functionally graded elastic material (Erdogan, 1985).

The solutions of the singular integral equation with the Cauchy type kernel have the form

$$g_i(t) = \frac{G_i(t)}{\sqrt{(t-a)(b-t)}}, \quad i = 1,2 \tag{25}$$

where $G_i(t)$ are bounded functions. The stress intensity factors and electric displacement intensity factors can be derived (Muskhhelishvili, 1953). The results are:

$$k_3(b) = \lim_{x\to b^+} \sqrt{2(x-b)}\tau_{yz(1)}(x,0) = -c_0 \mathrm{e}^{\beta b}\frac{G_1(b)}{\sqrt{(b-a)/2}} - e_0 \mathrm{e}^{\beta b}\frac{G_2(b)}{\sqrt{(b-a)/2}} \tag{26}$$

$$k_3(a) = \lim_{x\to a^-} \sqrt{2(a-x)}\tau_{yz(1)}(x,0) = c_0 \mathrm{e}^{\beta a}\frac{G_1(a)}{\sqrt{(b-a)/2}} + e_0 \mathrm{e}^{\beta a}\frac{G_2(a)}{\sqrt{(b-a)/2}} \tag{27}$$

$$k_3^D(b) = \lim_{x\to b^+} \sqrt{2(x-b)}D_{y(1)}(x,0) = -e_0 \mathrm{e}^{\beta b}\frac{G_1(b)}{\sqrt{(b-a)/2}} + \varepsilon_0 \mathrm{e}^{\beta b}\frac{G_2(b)}{\sqrt{(b-a)/2}} \tag{28}$$

$$k_3^D(a) = \lim_{x\to a^-} \sqrt{2(a-x)}D_{y(1)}(x,0) = e_0 \mathrm{e}^{\beta a}\frac{G_1(a)}{\sqrt{(b-a)/2}} - \varepsilon_0 \mathrm{e}^{\beta a}\frac{G_2(a)}{\sqrt{(b-a)/2}} \tag{29}$$

In order to obtain the specific functions $G_i(a)$ and $G_i(b)$ ($i = 1,2$), Eqs. (22), (23) and conditions (11) are normalized as (Erdogan et al., 1973):

$$\begin{aligned}\tau_{yz(1)}(x,0) = {} & c_0 \mathrm{e}^{\beta(a_0\bar{x}+c)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\bar{t}-\bar{x}} + \int_{-1}^{1}(h_1(a_0\bar{x}+c, a_0\bar{t}+c) + k_2(a_0\bar{x}+c, a_0\bar{t}+c))\right] f_1(\bar{t})\,\mathrm{d}\bar{t} \\ & + e_0 \mathrm{e}^{\beta(a_0\bar{x}+c)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\bar{t}-\bar{x}} + \int_{-1}^{1}(h_1(a_0\bar{x}+c, a_0\bar{t}+c) + k_2(a_0\bar{x}+c, a_0\bar{t}+c))\right] f_2(\bar{t})\,\mathrm{d}\bar{t}\end{aligned} \tag{30}$$

$$\begin{aligned}D_{y(1)}(x,0) = {} & e_0 \mathrm{e}^{\beta(a_0\bar{x}+c)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\bar{t}-\bar{x}} + \int_{-1}^{1}(h_1(a_0\bar{x}+c, a_0\bar{t}+c) + k_2(a_0\bar{x}+c, a_0\bar{t}+c))\right] f_1(\bar{t})\,\mathrm{d}\bar{t} \\ & - \varepsilon_0 \mathrm{e}^{\beta(a_0\bar{x}+c)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\bar{t}-\bar{x}} + \int_{-1}^{1}(h_1(a_0\bar{x}+c, a_0\bar{t}+c) + k_2(a_0\bar{x}+c, a_0\bar{t}+c))\right] f_2(\bar{t})\,\mathrm{d}\bar{t}\end{aligned} \tag{31}$$

$$\int_{-1}^{1} f_1(\bar{t})\,\mathrm{d}\bar{t} = \int_{-1}^{1} f_2(\bar{t})\,\mathrm{d}\bar{t} = 0 \tag{32}$$

The dimensionless length $\bar{x}$ and $\bar{t}$ are defined by

$$\bar{x} = \frac{x-c}{a_0}, \quad \bar{t} = \frac{t-c}{a_0} \tag{33a}$$

$$f_1(\bar{t}) = g_1(t), \quad f_2(\bar{t}) = g_2(t) \tag{33b}$$

Eqs. (30) and (31) are the singular integral equation of the first kind. It can be solved numerically by Gauss–Chebyshev integration formula (Erdogan et al., 1973). Because the solution has integrable singularities at both ends of the crack, the index of the problem $\kappa = 1$ defined in Erdogan et al. (1973). Thus the relationship between fundamental and weighting function is

$$f_1(\bar{t}) = \frac{F_1(\bar{t})}{\sqrt{(1+\bar{t})(1-\bar{t})}}, \quad f_2(\bar{t}) = \frac{F_2(\bar{t})}{\sqrt{(1+\bar{t})(1-\bar{t})}} \tag{34}$$

Eqs. (30)–(32) can be solved after reducing them into the following Chebyshev polynomial:

$$\begin{cases} \tau(x_r) = c_0 e^{\beta(a_0x_r+c)} \sum_{k=1}^{n} \frac{1}{n} F_1(t_k)\left[\frac{1}{t_k - x_r} + \pi(h_1(a_0x_r+c, a_0t_k+c) + k_2(a_0x_r+c, a_0t_k+c))\right] \\ \qquad + e_0 e^{\beta(a_0x_r+c)} \sum_{k=1}^{n} \frac{1}{n} F_2(t_k)\left[\frac{1}{t_k - x_r} + \pi(h_1(a_0x_r+c, a_0t_k+c) + k_2(a_0x_r+c, a_0t_k+c))\right] \\ D(x_r) = e_0 e^{\beta(a_0x_r+c)} \sum_{k=1}^{n} \frac{1}{n} F_1(t_k)\left[\frac{1}{t_k - x_r} + \pi(h_1(a_0x_r+c, a_0t_k+c) + k_2(a_0x_r+c, a_0t_k+c))\right] \\ \qquad - \varepsilon_0 e^{\beta(a_0x_r+c)} \sum_{k=1}^{n} \frac{1}{n} F_2(t_k)\left[\frac{1}{t_k - x_r} + \pi(h_1(a_0x_r+c, a_0t_k+c) + k_2(a_0x_r+c, a_0t_k+c))\right] \\ \sum_{k=1}^{n} \frac{\pi}{n} F_1(t_k) = 0 \\ \sum_{k=1}^{n} \frac{\pi}{n} F_2(t_k) = 0 \end{cases} \tag{35}$$

where $t_k = \cos\frac{(2k-1)}{2n}\pi$, $k = 1, 2, \ldots, n$; $x_r = \cos\frac{r}{n}\pi$, $r = 1, 2, \ldots, n-1$ are the nodes satisfy Chebyshev polynomial of the first and second kind respectively (Rivlin, 1974). According to the relationship between Eqs. (25), (33) and (34), the intensity factors can be expressed as different forms as follows:

$$k_3(b) = -c_0 e^{\beta b}\sqrt{a_0}F_1(1) - e_0 e^{\beta b}\sqrt{a_0}F_2(1) \tag{36}$$

$$k_3(a) = c_0 e^{\beta a}\sqrt{a_0}F_1(-1) + e_0 e^{\beta a}\sqrt{a_0}F_2(-1) \tag{37}$$

$$k_3^D(b) = -e_0 e^{\beta b}\sqrt{a_0}F_1(1) + \varepsilon_0 e^{\beta b}\sqrt{a_0}F_2(1) \tag{38}$$

$$k_3^D(a) = e_0 e^{\beta a}\sqrt{a_0}F_1(-1) - \varepsilon_0 e^{\beta a}\sqrt{a_0}F_2(-1) \tag{39}$$

Here the unknown values of $F_i(-1)$ and $F_i(1)$, $(i = 1, 2)$ can be obtained from the quadratic extrapolation from $F_i(t_n - 1)$, $F_i(t_n - 2)$, $F_i(t_n - 3)$ and $F_i(t_2)$, $F_i(t_3)$, $F_i(t_4)$, respectively.

2.2. Permeable crack problem

In this case, the conditions (8f) and (8j) of Section 2.1 should be modified to:

$$\phi_1(x, 0) = 0 \quad \text{for } 0 \leqq x < \infty \tag{40a}$$

$$D_{y(1)}(x, 0) = D_c(x, 0) = D(x) \quad \text{for } a < x < b \tag{40b}$$

where $D_c(x,0)$ denotes the electric displacement of the space of the crack itself. To satisfy the permeable condition, we need only one dislocation function $g_1(x)$. Following similar procedures of previous Section 2.1, the corresponding stress and electric displacement can be expressed as

$$\tau_{yz(1)}(x, 0) = \tau(x) = c_0 e^{\beta x}\frac{1}{\pi}\int_a^b \left[\frac{1}{t - x} + h_1(x, t) + k_2(x, t)\right] g_1(t)\,dt \tag{41}$$

$$D_{y(1)}(x, 0) = D(x) = e_0 e^{\beta x}\frac{1}{\pi}\int_a^b \left[\frac{1}{t - x} + h_1(x, t) + k_2(x, t)\right] g_1(t)\,dt \tag{42}$$

Therefore the stress intensity factor k_3 and the electric displacement intensity factor k_3^D are:

$$k_3(b) = -c_0 e^{\beta b}\sqrt{a_0}F_1(1) \tag{43}$$

$$k_3(a) = c_0 e^{\beta a}\sqrt{a_0}F_1(-1) \tag{44}$$

$$k_3^D(b) = -e_0 e^{\beta b}\sqrt{a_0}F_1(1) \tag{45}$$

$$k_3^D(a) = e_0 e^{\beta a}\sqrt{a_0}F_1(-1) \tag{46}$$

Since the crack is assumed to be electrically permeable, the condition (40a) results in the fact that the electric field E_y is continuous across the crack surfaces and remains in a finite value at the crack tips. However, from the constitutive equations of piezoelectric material, the electrical displacement D_y is related to the shear strain γ_{yz} and the piezoelectric constant e_{15}. Therefore, D_y must be singular at the crack tips due to the discontinuous displacement of the crack surface. The corresponding electrical displacement intensity factors k_3^D thus depend only on the material constant e_0 and not on the applied electric load.

The electric displacement intensity factor can be obtained by the relationship (Li and Tang, 2002; Wang, 2003)

$$k_3^D(i) = \frac{e_0}{c_0}k_3(i), \quad i = a, b \tag{47}$$

If the piezoelectric constant e_0 is imposed to be zero, the results can be reduced to the case of functionally graded elastic material (Erdogan, 1985).

3. Five degenerated problems

The crack problem we have discussed can be reduced to five degenerated problems. The crack is parallel to the direction of material gradient. To the authors' knowledge, these simple problems have not been solved yet.

3.1. Problem A: a FGPM half space contains a crack normal to the free surface $x = 0$ (i.e. $\gamma \to \infty$)

In this case, the material 2 is removed away by setting the material parameter γ approaches infinity. The boundary conditions on the free surface $x = 0$ become traction free and electrically opened. If we replace conditions (8) by $\tau_{xz}(0, y) = 0$ and $D_x(0, y) = 0$ and solve the problem from the beginning, Eqs. (17) and (18) will be the results. The kernel k_1 is unchanged and the function $K_2(\alpha)$ in Eq. (21) should be replaced by

$$K_2(\alpha) = \frac{\alpha^2}{\alpha_1\left(\alpha_1 + \frac{\beta}{2}\right)} \tag{48}$$

This result can also be obtained from Eq. (21) when $\gamma \to \infty$.

3.2. Problem B: a FGPM half space contains a crack normal to a rigid surface $x = 0$ (i.e. $\gamma \to -\infty$)

If the material parameter γ approaches minus infinity, the material 2 is assumed to be rigid. This is the case of an elastic half space bonded to a rigid half space. The boundary conditions on the rigid surface $x = 0$ is fixed in displacement and is electrically closed, i.e. $w(0, y) = \phi(0, y) = 0$. If the conditions (8) are placed, the singularity integral equations are again the Eqs. (17) and (18). The kernel k_1 is unchanged and the function $K_2(\alpha)$ in Eq. (21) should be replaced by

$$K_2(\alpha) = \frac{-\alpha^2}{\alpha_1\left(\alpha_1 - \frac{\beta}{2}\right)} \tag{49}$$

This result can also be obtained from Eq. (21) when $\gamma \to -\infty$.

3.3. Problem C: a homogeneous piezoelectric half space contains a crack and is bonded to a FGPM half space ($\beta = 0$)

If the material 1 is a homogeneous piezoelectric half space, the function $K_2(\alpha)$ becomes as follows:

$$K_2(\alpha) = \frac{\left(\alpha - \alpha_2 + \frac{\gamma}{2}\right)}{\left(\alpha_2 + \alpha - \frac{\gamma}{2}\right)} \tag{50}$$

3.4. Problem D: a FGPM half space contains a crack and is bonded to a homogeneous piezoelectric half space ($\gamma = 0$)

If the material 2 is a homogeneous piezoelectric half space, the function $K_2(\alpha)$ becomes as follows:

$$K_2(\alpha) = \frac{\alpha^2\left(\alpha_1 - \alpha - \frac{\beta}{2}\right)}{\alpha_1\left(\alpha + \alpha_1 + \frac{\beta}{2}\right)\left(\alpha_1 - \frac{\beta}{2}\right)} \tag{51}$$

3.5. General problem: an infinite FGPM medium ($\gamma = \beta$) or two different FGPM half planes ($\gamma = \beta$) contains an internal crack

If the material parameters γ and β are equal, there is no interface. The function $K_2(\alpha)$ or k_2 vanishes.

4. Results and discussions

In the following discussions, we take PZT-4 as the base material. The material properties are as follows:

$$c_0 = 25.6\,\text{GPa}, \quad e_0 = 12.7\,\text{C/m}^2, \quad \varepsilon_0 = 6.46 \times 10^{-9}\,\text{C/Vm}$$

The variations of functionally graded piezoelectric material properties are in the exponential forms of (2). For convenience, the stress and electric displacement intensity factors are normalized as

$$k_i = \frac{k_3(i)}{\tau_0\sqrt{a_0}} = \frac{k_3^D(i)}{D_0\sqrt{a_0}}, \quad i = a, b \tag{52}$$

In Eq. (35), we use $\tau_0 = 4.2\,\text{MPa}$ and $D_0 = 0.002\,\text{C/m}^2$, which are the uniform shear stress and electric displacement applied on the crack surfaces (Pak, 1990). Firstly, we discuss the degenerated problems.

4.1. Degenerated problem A: $\gamma \to \infty$

This case is a FGPM half space contains a crack normal to the surface $x = 0$, which is traction free and electrically opened. Fig. 2 plots the variations of the normalized intensity factors at crack tips (i.e. k_a and k_b) with normalized nonhomogeneous parameter βa_0 at different values of c/a_0 when $a_0 = 2/3\,\text{cm}$. It shows that greater normalized intensity factors occur at the crack tip with stronger material properties. Fig. 2 can be explained in detail by looking at Fig. 3, which shows the effects of the material property variations β on

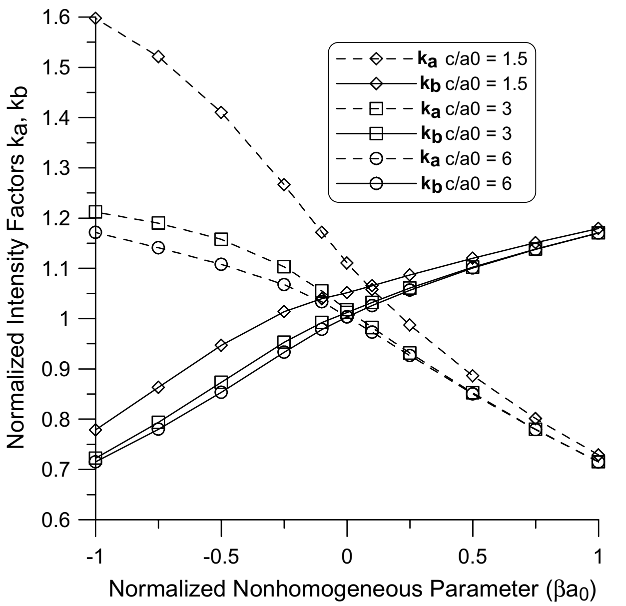

Fig. 2. Variations of the normalized intensity factors at crack tips *a* and *b* with normalized nonhomogeneous parameter βa_0 at different values of c/a_0 when $a_0 = 2/3\,\text{cm}$ ($\gamma \to \infty$).

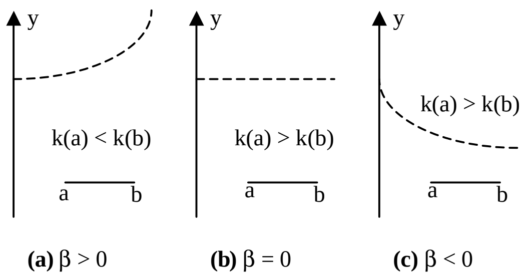

Fig. 3. The effects of the material property variations on the intensity factors (surface $x = 0$ is traction free and electrically opened).

the intensity factors. The material properties on the crack tip side *a* are stronger when β is negative, and vice versa. Consequently, the stresses and the electrical displacements and their related intensity factors become higher at crack tip with higher material properties. For the homogeneous case with $\beta = 0$, the intensity factors at crack tip *a* should be greater than those near the crack tip *b*. If the crack is close to the surface $x = 0$, the edge effect appears apparently. In order to get equal intensity factors at both crack tips (i.e. $k_a = k_b$), the material properties near the crack tip *b* must be raised by a small amount value of β. This phenomena can be observed from Fig. 2 that $k_a \fallingdotseq k_b$ when $\beta = 0$, and $c/a_0 = 3$ or 6. However, if the crack is closer to the surface $x = 0$ (say, $c/a_0 = 1.5$), the lines k_a and k_b intersect at the point $k_a = k_b = 1.06435$ when $\beta a_0 \fallingdotseq 0.091054$.

4.2. Degenerated problem B: $\gamma \to -\infty$

This case is a FGPM half space contains a crack normal to the surface $x = 0$, which is clamped and electrically closed. Fig. 4 plots the variations of the normalized intensity factors at crack tips (i.e. k_a and k_b)

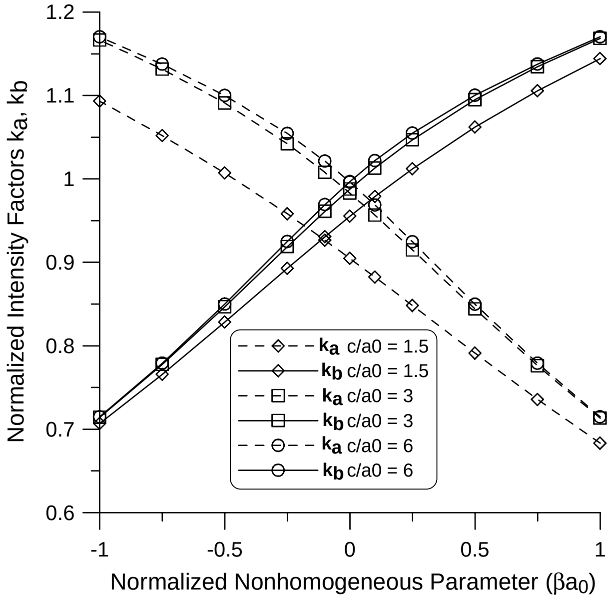

Fig. 4. Variations of the normalized intensity factors at crack tips *a* and *b* with normalized nonhomogeneous parameter βa_0 at different values of c/a_0 when $a_0 = 2/3\,\text{cm}$ ($\gamma \to -\infty$).

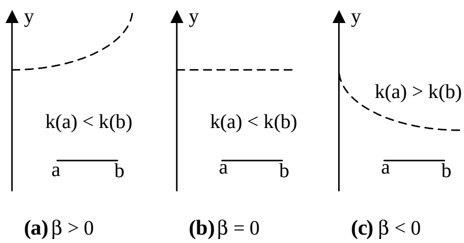

Fig. 5. The effects of the material property variations on the intensity factors (surface $x = 0$ is clamped and electrically closed).

with normalized nonhomogeneous parameter βa_0 at different values of c/a_0 when $a_0 = 2/3\,\text{cm}$. Again, the results show that greater normalized intensity factors occur at the crack tip with stronger material properties. Fig. 5 shows the effects of the material property variations on the intensity factors of this case $\gamma \to -\infty$. For the homogeneous case with $\beta = 0$, the intensity factors at crack tip *a* should be smaller than those near the crack tip *b*. In order to get equal intensity factors at both crack tips (i.e. $k_a = k_b$), the material properties near the crack tip *b* must be decreased by a small amount value of β. This phenomena can also be observed from Fig. 4 that $k_a \fallingdotseq k_b$ when $\beta = 0$, and $c/a_0 = 3$ or 6. However, if the crack is closer to the surface $x = 0$ (say, $c/a_0 = 1.5$), the lines k_a and k_b intersect at the point $k_a = k_b = 0.928167$ when $\beta a_0 \fallingdotseq -0.109212$.

4.3. Degenerated problem C: $\beta = 0$

In this case, the material 1 is reduced to a homogeneous piezoelectric material with $c_0 = 25.6\,\text{GPa}$, $e_0 = 12.7\,\text{C/m}^2$, and $\varepsilon_0 = 6.46 \times 10^{-9}\,\text{C/Vm}$. Fig. 6 shows the variations of normalized intensity factors with material parameter γ of material 2 when $a_0 = 2/3\,\text{cm}$. For the case $\gamma > 0$, the material properties of material

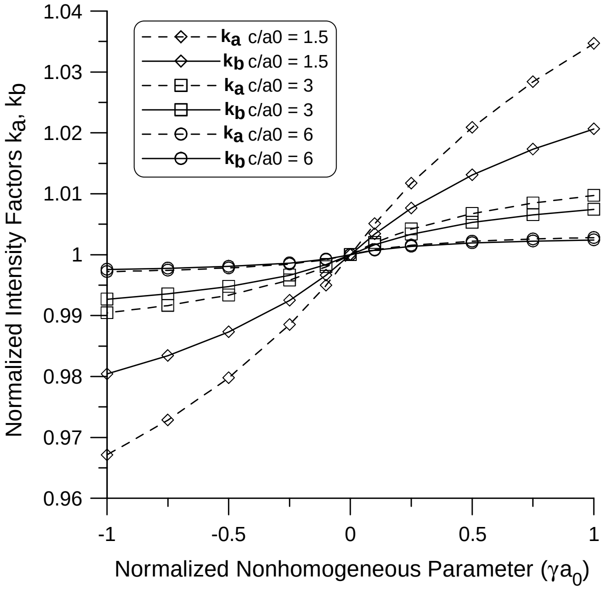

Fig. 6. Variation of normalized intensity factors with γ at different crack location ($\beta = 0$).

2 are weaker than those of the homogeneous material 1. It is expected that the intensity factors at crack tip *a* are greater than those at crack tip *b*. As the crack move away further from the interface, say from $c/a_0 = 1.5$ to 6, the magnitudes of the factors at crack tips *a* and *b* will approach same values. For the other case with $\gamma < 0$, the material properties of material 2 are stronger. The intensity factors at crack tip *b*

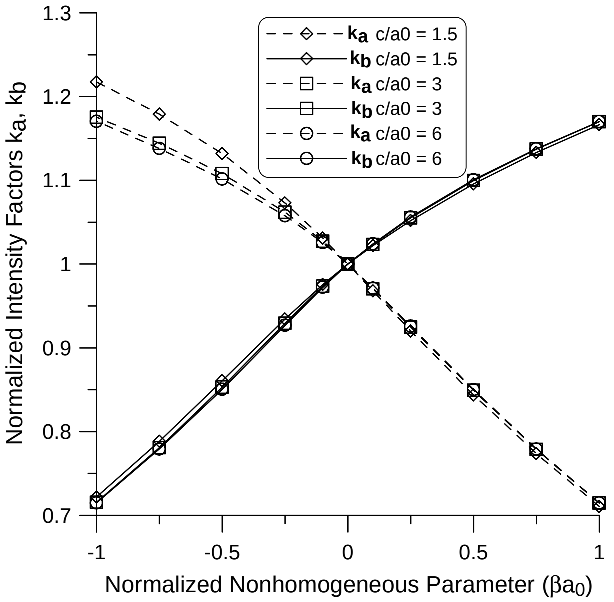

Fig. 7. Variation of normalized intensity factors with β at different crack location ($\gamma = 0$).

become larger. Again, the magnitudes of the factors at crack tips *a* and *b* will gradually approach same values when the crack is far away from the interface. If $\beta = \gamma = 0$, the problem becomes a crack lies in an infinity homogeneous piezoelectric material. The normalized intensity factors at crack tips *a* and *b* are equal to one, which matches the conventional piezoelectric crack problem.

4.4. Degenerated problem D: $\gamma = 0$

Now, the material 2 is changed to a homogeneous piezoelectric material with $c_0 = 25.6\,\text{GPa}$, $e_0 = 12.7\,\text{C/m}^2$, and $\varepsilon_0 = 6.46 \times 10^{-9}\,\text{C/Vm}$. The normalized intensity factors are plotted in Fig. 7. Similar to

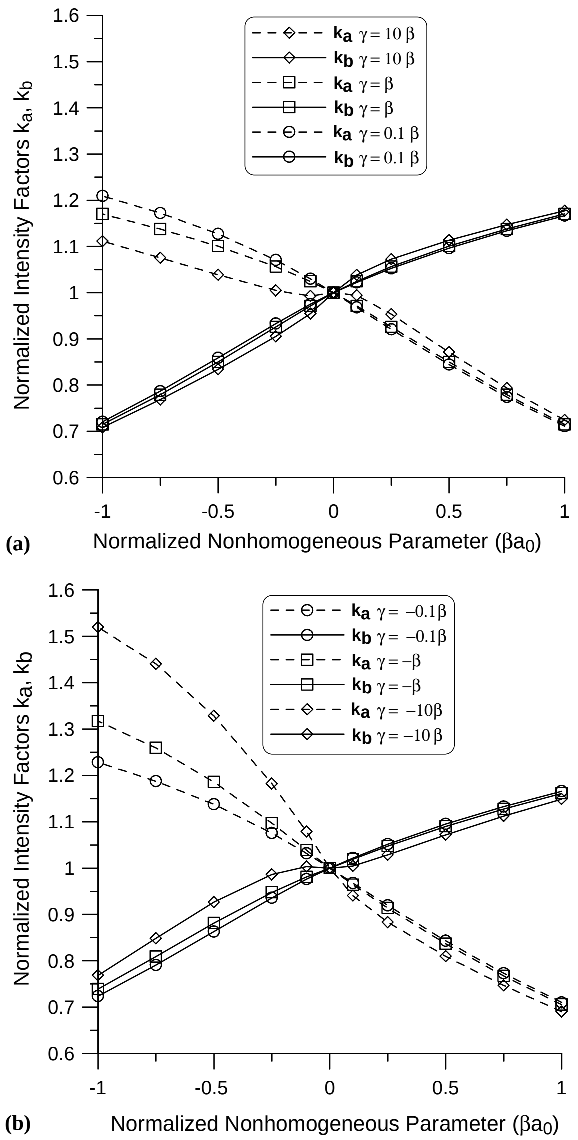

Fig. 8. Variation of the normalized intensity factors with material parameters β and γ. (a) $\gamma = 10\beta$, β, 0.1β; (b) $\gamma = -0.1\beta$, $-\beta$, -10β.

the degenerated problems A and B, the results show that greater normalized intensity factors occur at the crack tip with stronger material properties.

4.5. General problems: β and γ are finite and $\beta \neq \gamma \neq 0$.

Finally, we discuss the most general cases that materials 1 and 2 are two different functionally graded piezoelectric materials. Fig. 8(a) and (b) plot the variation of the normalized intensity factors with β and

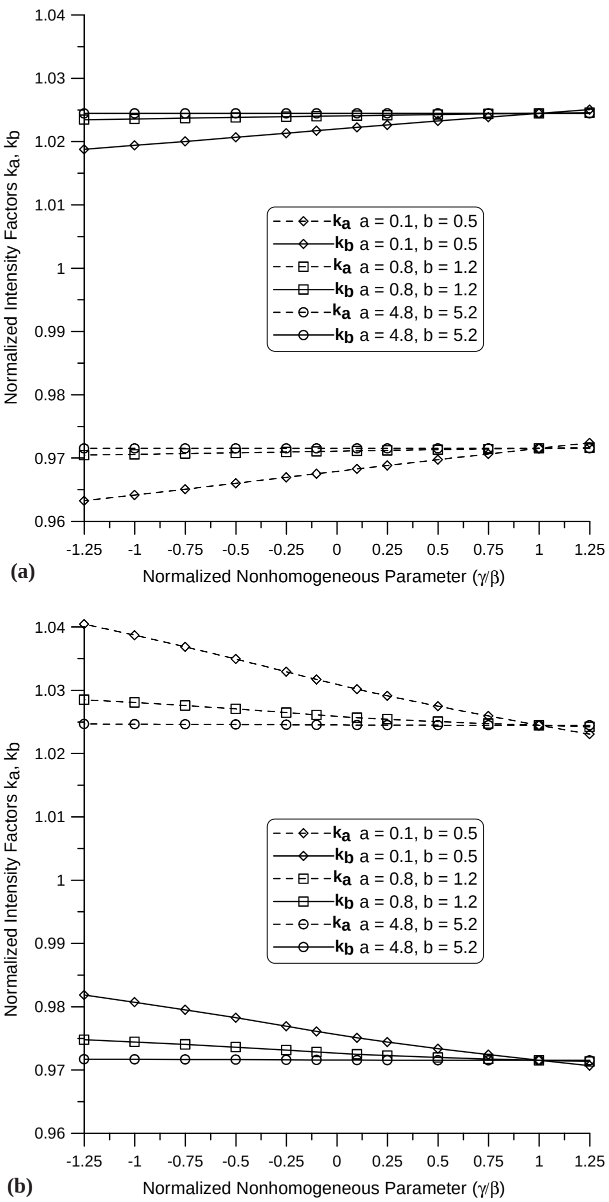

Fig. 9. Variation of the normalized intensity factors with constant crack length at different crack location. (a) $\beta = 0.5$; (b) $\beta = -0.5$.

γ when $a_0 = 2/3$ cm. We can see in general from the figure that the intensity factors are greater at the tip where the material properties are stronger. Two other results are: (i) For $\beta < 0$ and $\gamma < 0$ (Fig. 8(a)), the intensity factors for larger $|\gamma|$(say, $\gamma = 10\beta$) are smaller since the material properties of material 2 are stronger; and (ii) For $\beta < 0$ and $\gamma > 0$ (Fig. 8(b)), the intensity factors for larger γ (say, $\gamma = -10\beta$) are larger since the material properties of material 2 are weaker.

Fig. 9(a) and (b) plots the variations of normalized intensity factors with γ when $\beta = 0.5$ and -0.5, respectively. For $\beta = 0.5$, the material properties near the crack tip b are stronger. It can be seen clearly from Fig. 9(a) that the intensity factors are larger at crack tip b. For the case $\beta = \gamma$, i.e. the uniform functionally graded piezoelectric material, the normalized intensity factors are independent of the crack length. They are $k_a = 0.97154$ and $k_b = 1.02447$ for $\beta = 0.5$ and $k_a = 1.02447$ and $k_b = 0.97154$ for $\beta = -0.5$, respectively. It is also found that the normalized intensity factors k_a and k_b are nearly the same regardless the value of γ as the crack is far away from the interface.

For the permeable crack, the electric potential is continuous in the full region $0 \leqq x < \infty$. It is assumed the electric displacement is also continuous across the crack surface and the crack itself. Thus the electric field has no singularity at crack tips a and b and the stress intensity factors depend only on the exerted shear stress at the crack surface. It has been validated that the normalized stress intensity factors are the same as those for the elastic medium in the symmetric loading case (i.e. impermeable crack) (Chen and Yu, 1997). Thus the stress intensity factors for the permeable crack are also the same as the impermeable case (Li and Duan, 2001).

5. Conclusions

The fracture behavior of the internal crack located within one of two bonded functionally graded piezoelectric materials has been studied. Both the impermeable and permeable cases are considered. Under the antiplane shear and in plane electric displacement, the problem is reduced to a set of singular integral equations. The stress and electric displacement intensity factors for impermeable and permeable cracks are obtained by using Gauss–Chebyshev integration technique. The problem can be reduced to five degenerated simple cases. The results show that the normalized intensity factors are greater at the crack tip where the material properties are stronger. For the impermeable crack, the stress and electric displacement intensity factors depend on the applied mechanical and electric loads. However, the intensity factors for the permeable crack depend only on the mechanical loads.

References

Asfar, A.M., Sekine, H., 2000. Crack spacing effect on the brittle fracture characteristics of semi-infinite functionally graded materials with periodic edge cracks. International Journal of Fracture 102, L61–L66.

Chen, Z.T., Yu, S.W., 1997. Antiplane shear problem for a crack between two dissimilar piezoelectric materials. International Journal of Fracture 86, L9–L12.

Choi, H.J., 1996. Bonded dissimilar strips with a crack perpendicular to the functionally graded interface. International Journal of Solid and Structures 33, 4101–4117.

Delale, F., Erdogan, F., 1983. The crack problem for a nonhomogeneous plane. Transaction of the ASME, Journal of Applied Mechanics 50, 609–614.

Erdogan, F., Gupta, G.D., Cook, T.S., 1973. Numerical solution of singular integral equations. In: Sih, G.C. (Ed.), Mechanics of Fracture 1: Method of analysis and solution of crack problem. Noordhoff International Publishing, Leyden, The Netherlands, Chapter 7.

Erdogan, F., 1985. The crack problem for bonded nonhomogeneous materials under antiplane shear loading. Transactions of the ASME, Journal of Applied Mechanics 52, 823–828.

Erdogan, F., Kaya, A.C., Joseph, P.F., 1991a. The crack problem in bonded nonhomogeneous materials. Transaction of the ASME, Journal of Applied Mechanics 58, 410–418.

Erdogan, F., Kaya, A.C., Joseph, P.F., 1991b. The mode III crack problem in bonded materials with a nonhomogeneous interfacial zone. Transactions of the ASME, Journal of Applied Mechanics 58, 419–427.

He, L.H., Ye, R.Q., 2000. Concentration of electric field near electrodes on piezoelectric layer. Theoretical and Applied Fracture Mechanics 33, 101–106.

Li, C., Weng, G.J., 2002. Antiplane crack problem in functionally graded piezoelectric materials. Transactions of the ASME, Journal of Applied Mechanics 69, 481–488.

Li, X.F., Duan, X.Y., 2001. Closed-form solution for a mode-III crack at the mid-plane of a piezoelectric layer. Mechanics Research Communications 28, 703–710.

Li, X.F., Tang, G.J., 2002. Antiplane permeable edge cracks in a piezoelectric strip of finite width. International Journal of Fracture 118, L45–L50.

Muskhhelishvili, N.I., 1953. Singular Integral Equations. Noordhoff International Publishing, Groningen, The Netherlands.

Pak, Y.E., 1990. Crack extension force in a piezoelectric material. Transaction of the ASME, Journal of Applied Mechanics 57, 647–653.

Rivlin, T.J., 1974. The Chebyshev Polynomials. Wiley, New York.

Tomio, O., 1990. Optical beam deflection using a piezoelectric bimorph actuator. Sensors and Actuators A21–A23, 726–728.

Ueda, S., 2003. Crack in functionally graded piezoelectric strip bonded to elastic surface layers under electromechanical loading. Theoretical and Applied Fracture Mechanics 40, 225–236.

Wang, B.L., 2003. A mode III crack in functionally graded piezoelectric materials. Mechanics Research Communications 30, 151–159.

Wu, C.C.M., Kahn, M., Moy, W., 1996. Piezoelectric ceramics with functional gradients: a new application in material design. Journal of American Ceramics Society 79, 809–812.

Zhu, X., Wang, Q., Meng, Z., 1995. A functionally gradient piezoelectric actuator prepared by powder metallurgical process in PNN-PZ-PT system. Journal of Materials Science Letters 14, 516–518.